Lesson 9-4: Vectors

Getting from here to there

Let's say you live 5 miles from school in Lynnwood. Suppose you are trying to tell a friend how to get to your house from school. You tell your friend "to get to my house, go 5 miles." Hmm, not very helpful is it? Very likely your friend is going to say something like "Ah hello! Which direction???"

To get someplace, you need more than distance; you need to know direction. In math and physics this is called a **vector**.

Vectors

A vector has both direction and distance. The distance can be thought of as the vector's size or **magnitude**. You can draw a vector as an arrow pointed in the desired direction whose length is the vector's magnitude. The initial point of the vector is often called its tail and the terminal point of the vector is often called its head. If the tail is at point *K* and the head is at point *M*, we can label the vector $\overline{KM}$. Notice that while it looks like a ray, there is a key difference in the notation. The arrow for a vector is ½ barbed ($\rightharpoonup$) and the arrow for a ray is full barbed (→).

What are vectors used for?

Vectors are very useful for describing and working with anything that has magnitude and direction. They are used a lot in physics when working with force or velocity.

Have you ever gone swimming in a river? How about at the ocean beach (what about one with a rip tide???)? In either case, if you tried to just float, the current was pushing on you and moving you; the current was exerting a force on your body. This force had a magnitude (how hard it was pushing on you) and a direction. The current could be represented by a vector.

In the navy we used vectors all the time. For instance, if we were sailing from San Diego to Hawaii (yeah!) we would have to factor in the force of the ocean currents and wind on the ship. They would push us off course. We used vector math to help us figure out what direction and speed (magnitude) we needed to go to compensate for the currents and wind. If we didn't, we would completely miss Hawaii and hit (perhaps) China!

Ways of describing vectors

There are two common ways of describing a vector. The first is using (x, y) ordered pairs in the coordinate plane (coordinate geometry), and the second is with compass directions. Both are useful depending on what your situation is.

Describing vectors using coordinate geometry

Pick a point on the coordinate plane. Any point. For instance, I'll pick (3, 4). This point represents a vector. How can it you ask? Simple…this point represents a distance (magnitude) from the origin in a specific direction (go 3 units in the *x* direction and 4 units in the *y* direction). When using a point as a vector, we use special angle brackets instead of the parenthesis: $\langle 3,4\rangle$. What is the magnitude of the vector $\langle 3,4\rangle$? Use the distance formula to find out:

$$d=\sqrt{(3^2-0^2)+(4^2-0^2)}=\sqrt{9+16}=\sqrt{25}=5.$$

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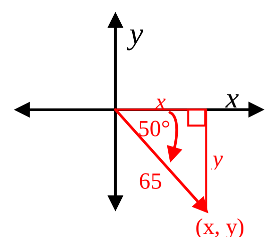
So a question: any time you use a coordinate point to come up with the description of a vector, where is the tail? It is at the origin. To use coordinate geometry to describe a vector, you determine the coordinate of the vector's head **when its tail is at the origin**. Now, one neat thing about vectors is that no matter where you move it, as long as you don't change its direction or magnitude, it is the same vector. Our $\langle 3,4\rangle$ vector is still the $\langle 3,4\rangle$ vector if you move it somewhere else.

One way to think about this is to realize the following. If the tail of a vector is at the origin, the coordinate of the head represents the slope of the vector! The *y* coordinate tells you how far to move on the *y*-axis (rise) and the *x* coordinate tells you how far to move on the *x*-axis (run). The slope doesn't change as long as you don't change the direction the vector points. Thus you can move it all over and it is still the same vector.

The trig tie

Now, suppose you have a vector for which we know the magnitude (65 miles) and direction (a 50º angle down from the x-axis). Using this information, can you describe the vector as an ordered pair? Here are a couple of hints: can you make a right triangle that includes this vector so it is the hypotenuse of the triangle? To form the triangle, use the axis the angle is measured from.

Here is the diagram with the triangle. Now, what information do we have? We know an angle measure and the length of the hypotenuse. We want to determine the coordinate of the head of the vector. The *x* value is the length of the adjacent side and the *y* value is the length of the opposite side. What trig functions will give us those values? Cosine for *x* (adjacent over hypotenuse) and sine for *y* (opposite over hypotenuse):



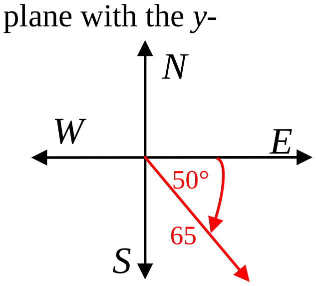
$$\cos 50 = \frac{x}{65}; x = 65\cos 50 = 41.8 \textit{ and } \sin 50 = \frac{y}{65}; y = 65\sin 50 = 49.8$$

So our vector (65 miles at 50º angle down from the x-axis) can also be described by the ordered pair $\langle 41.8, 49.8\rangle$.

Describing vectors using compass directions

Now, would you give directions to your friend using a point on the map? Using our example from above would you say "go to point $\langle 41.8, 49.8\rangle$." Ah…no. You'd most likely use compass directions: "go 41.8 miles east then 49.8 miles south." That would be much more helpful! But wouldn't it be *even* more helpful to just tell them to go straight there? Our problem above states the direction as "a 50º angle down from the x-axis". Rats! That isn't helpful at all! Look outside; where the heck is the *x*-axis???

Notice that a compass rose is basically the same thing as the coordinate plane with the *y*-axis running north (up) and south (down), the *x*-axis running east (right) and west (left). Given this, we have a nice way of describing our vector. Our diagram above can be redrawn as follows. Using compass directions, we can then describe our vector as "65 miles 50° south of east."



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Now, using compass directions again, can you think of another way of describing this vector? Consider the complement of the 50° angle. How about "65 miles 40° east of south"? Both are valid; they both describe the same vector. The description you use is best chosen to match the problem you are working with. For instance, if we were told "go due east and then move 50° south" it would make most sense to describe the resulting vector as "65 miles 50° south of east."

Vector notation

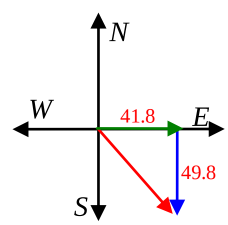
A common way of naming a vector is using a lower case letter with a ½ barb arrow over it: $\vec{w}$ would be the vector *w*.

So…here is our vector described in three different ways:

- $\vec{w} = \langle 41.8, 49.8 \rangle$
- 65 miles at a 50º angle down from the x-axis
- 65 miles 50° south of east (or 40° east of south)

Adding vectors

Remember the directions we gave a little bit ago? "Go 41.8 miles east then 49.8 miles south"? This results in a vector right? But getting there, aren't we using two other vectors? There is one whose direction is east with a magnitude of 41.8 and a second with direction south and magnitude 49.8. The vector that results from this trip can be thought of as adding the second vector to the first. In effect, we're putting the tail of the second at the head of the first and connecting the end points. Here is a picture with the red vector being the resulting vector. We call the resulting vector the **resultant**. If we called the first vector $\vec{u}$, the second $\vec{v}$ and the resultant $\vec{w}$, you'd write this as $\vec{w} = \vec{u} + \vec{v}$. To add vectors, all you do is add the corresponding coordinates of the heads together. Remember, when we describe vectors using coordinates, we determine the point when the vector's tail is at the origin.



$$\text{if } \vec{u} = \langle x_1, y_1 \rangle \text{ and } \vec{v} = \langle x_2, y_2 \rangle \text{ then } \vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2 \rangle$$

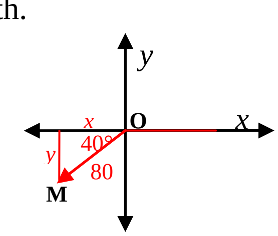
Using our example, if $\vec{u} = \langle 41.8, 0 \rangle$ *and* $\vec{v} = \langle 0, 49.8 \rangle$ then

$$\vec{u} + \vec{v} = \langle 41.8 + 0,\ 0 + 49.8 \rangle = \langle 41.8, 49.8 \rangle$$

Examples

1. Describe $\overline{OM}$ as an ordered pair. Give coordinates to the nearest tenth.

What do we know? Angle measure and hypotenuse, we want opposite and adjacent.



$$\cos 40 = \frac{x}{80};\ x = 80 \cos 40 = 61.28 \approx 61.3 \text{ (in the negative } x \text{ dir)}$$

$$\sin 40 = \frac{y}{80};\ y = 80 \sin 40 = 51.42 \approx 51.4 \text{ (in the negative } y \text{ dir)}$$

$$\overline{OM} = \langle -61.3, -51.4 \rangle$$

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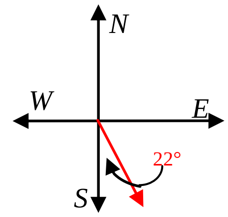
2. Use compass directions to describe the direction of the vector.

22° east of south.

or

68° south of east (but 1st is preferable because of the way the problem is stated)



3. A boat sailed 12 mi east and 9 mi south. The trip can be described by the vector $\langle 12, -9 \rangle$. Use distance and direction to describe this vector a second way.

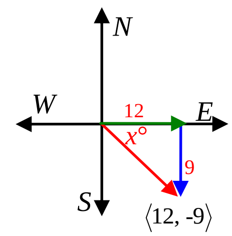
We want to find the angle the resultant makes with the *x*-axis.
We know the opposite side and adjacent side…tangent ratio.

$$x = \tan^{-1}\left(\frac{9}{12}\right) = 36.87 \approx 37°$$

Now the distance sailed: use the Pythagorean Theorem.

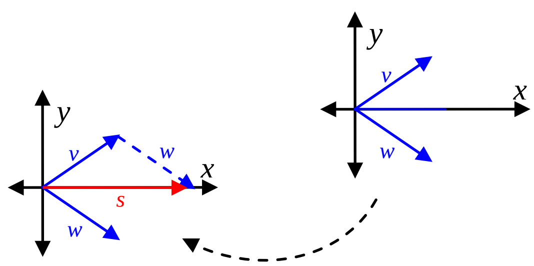
$$\sqrt{9^2 + 12^2} = 15$$

The boat sailed 15 miles at about 37° south of east.



4. Vectors $\vec{v}\langle 4,3\rangle$ and $\vec{w}\langle 4,-3\rangle$ are shown. Write $\vec{s}$, their sum, as an ordered pair.

$$\vec{s} = \vec{v} + \vec{w} = \langle 4+4, 3+{}^{-}3\rangle = \langle 8, 0\rangle$$



5. An airplane's speed is 250 *mph* in still air. The wind is blowing due east at 20 *mph*. If the airplane heads due north, what is its resultant speed and bearing (direction)? Round to the nearest unit. Diagram is not to scale.

The airplane's vector is $\langle 0, 250\rangle$

The wind's vector is $\langle 20, 0\rangle$

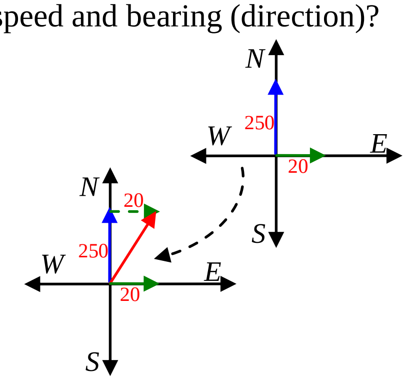
The resultant vector is $\langle 20, 250\rangle$

The resultant's speed (magnitude) is:

$$\sqrt{20^2 + 250^2} = 250.799 \approx 251 mi$$

The resultant's bearing is:

$$x° = \tan^{-1}\left(\frac{20}{250}\right) = 4.57 \approx 5°$$ … 251 *mi* at about 5° east of north.



Homework Assignment

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